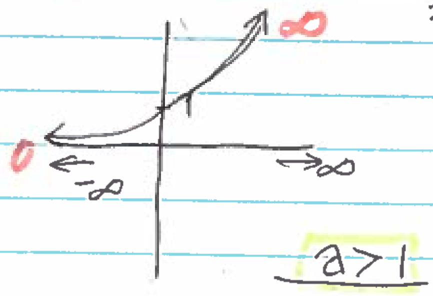
Ch 3.1 ⟨ Exponential Function ⟩

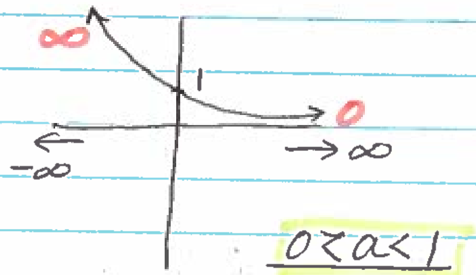
[1] $\boxed{a^x = \lim_{r \to x} a^r}$ r = rational

[2] If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain $\mathbb{R}$, and Range $(0, \infty)$.
In particular, $a^x > 0$ for all x.
If $a, b > 0$ and $x, y \in \mathbb{R}$ then

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = \dfrac{a^x}{a^y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

[3] If $a > 1$, then $\lim_{x \to \infty} a^x = \infty$ and $\lim_{x \to -\infty} a^x$



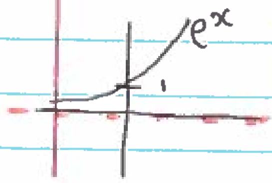


If $0 < a < 1$, then $\lim_{x \to \infty} a^x = 0$, $\lim_{x \to -\infty} a^x =$

[4] $\boxed{e = \lim_{x \to 0} (1 + x)^{1/x}}$ ≈ 2.718281828

[5] PROPERTIES OF NATURAL EXPONENTIAL FUNCTION

The exponential function $f(x) = e^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$.
Thus $e^x > 0$ for all x. Also



$\boxed{\lim_{x \to -\infty} e^x = 0}$, $\boxed{\lim_{x \to \infty} e^x = \infty}$

⟨Ch 3.2⟩ Inverse Function & Logarithm

$\boxed{1}$ Definition

A function f is called a one-to-one function if it never takes on the same value twice.

that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

★ Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

$\boxed{2}$ Definition

Let f be a one-to-one function with domain A and Range B.

Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

$\boxed{3}$ $$f^{-1}(x) = y \iff f(y) = x$$

$\boxed{4}$

$f^{-1}(f(x)) = x$ for every x in A

$f(f^{-1}(x)) = x$ for every x in B

$$f(x) = x^3$$
$$f^{-1}(x) = x^{1/3}$$
$$f(f^{-1}(x)) = f(x^{1/3})^3 = x$$
$$f^{-1}(f(x)) = f^{-1}(x^3)^{1/3} = x$$

⟨Ch 3.2⟩ Inverse Function & Logarithm

5 How to find the inverse function of a one-to-one Function f

① Write $y = f(x)$

② Solve this equation for x in terms of y (i

③ To express f^{-1} as a function of x, interchange x and y.
The resulting equation is $f^{-1}(x) = y$

6 Theorem

If f is a one-to-one continuous function defin on an interval, then its inverse function f^{-1} is also continuous.

7 Theorem

If f is a one-to-one differentiable function wit inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at and

$$\boxed{(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}}$$

☆ Proof

$$(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}$$

If $f(b) = a$ then $f^{-1}(a) = b$.
And if we let $y = f^{-1}(x)$, then $x = f(y)$.
Since f is differentiable, it is continuous, so f^{-1} i continuous by theorem 6.
Thus if $x \to a$, then $f^{-1}(x) \to f^{-1}(a)$, that is $y \to$
Therefore

$$f^{-1}(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} = \lim_{y \to b} \frac{y - b}{f(y) - f(b)}$$

$$= \lim_{y - b} \frac{1}{\frac{f(y) - f(b)}{y - b}} = \frac{1}{\lim_{y - b} \frac{f(y) - f(b)}{y - b}} = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a$$

8 $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

9 $$\boxed{\log_a x = y \iff a^y = x}$$

10 $$\boxed{\log_a(a^x) = x} \quad \text{for every } x \in \mathbb{R}$$

$$\boxed{a^{\log_a x} = x} \quad \text{for every } x > 0$$

☆ LAWS OF LOGARITHM

If x and y are positive numbers, then

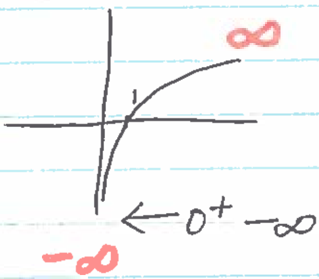
① $\log_a(xy) = \log_a x + \log_a y$

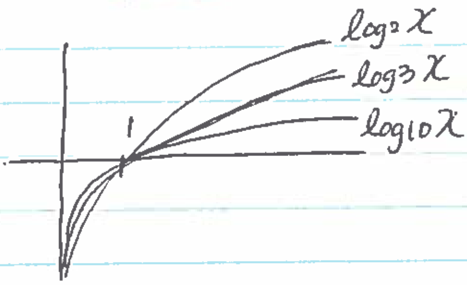
② $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

③ $\log_a(x^r) = r\log_a x$ (where r is $\mathbb{R}$)

11 If $a > 1$, then

$$\lim_{x\to\infty} \log_a x = \infty \quad \text{and} \quad \lim_{x\to 0^+} \log_a x = -\infty$$





☆ $\boxed{\log_e x = \ln x}$

12 $$\boxed{\ln x = y \iff e^y = x}$$

⟨3.2⟩ Inverse Function and Logarithm

13 $\ln(e^x) = x \qquad x \in \mathbb{R}$

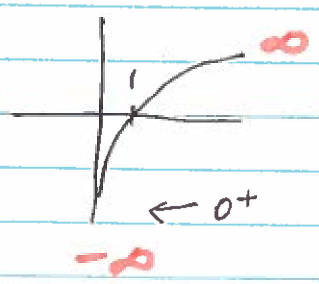
$e^{\ln x} = x \qquad x > 0$

☆ $\boxed{\ln e = 1}$

14 CHANGE OF BASE FORMULA

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

15 $\lim_{x \to \infty} \ln x = \infty \qquad \lim_{x \to 0^+} \ln x = -\infty$



⟨3.3⟩ Derivatives of logarithm & exponential functions

1 The function $f(x) = \log_a x$ is differentiable a

$$f'(x) = \frac{1}{x} \log_a e$$

2 $$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

3 DERIVATIVE OF NATURAL LOGARITHM FUNCTION

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

4 $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ or $\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$

$$\boxed{\frac{d}{dx}(\ln u) = \frac{u'}{u}}$$

5 $\frac{d}{dx}\ln|x| = \frac{1}{x}$

☆ STEPS IN LOGARITHMIC DIFFERENTIATION

1. Take natural logarithm of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate Implicitly with respect to x.
3. Solve the resulting equation for y'.

☆ THE POWER RULE

If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

6 Theorem

The exponential function $f(x) = a^x$, $a > 0$

~~we use the fact that exponential and logarithm~~

is differentiable and

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

7 DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

8 $$\boxed{\frac{d}{dx}(e^u) = e^u(u')}$$

⟨3.4⟩ EXPONENTIAL GROWTH AND DECAY

1. $\dfrac{dy}{dt} = ky$ law of natural growth
   law of natural decay

   differential equation

2. THEOREM

   The only solutions of the differential equati
   $dy/dt = ky$ ~~and~~ are the exponential fu

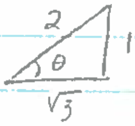
$$\boxed{y(t) = y(0)e^{kt}}$$

3. $\dfrac{dP}{dt} = kP$ or $\dfrac{1}{P}\dfrac{dP}{dt} = k$

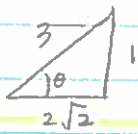
⟨3.5⟩ Inverse Trigonometric Functions

1. $\sin^{-1}x = y \iff \sin y = x, \quad -\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

   $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$



   $\tan\left(\arcsin\frac{1}{3}\right) = \dfrac{1}{2\sqrt{2}}$



2. $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

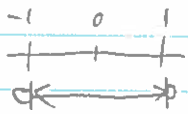
   Domain

   $\sin(\sin^{-1}) = x$ for $-1 \le x \le 1$

   Domain

3. $$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \boxed{-1 < x < 1}$$

$1 - x^2 > 0$
$-x^2 > -1$
$x^2 < 1$
$x < +1$



4. $\cos^{-1} x = y \iff \cos y = x$ and $0 \le y \le \pi$

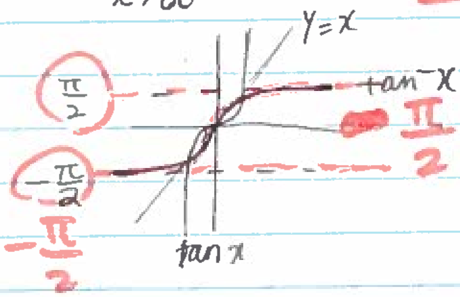
5. $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$

$\cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$

6. $\boxed{\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}}$ $\boxed{-1 < x < 1}$

7. $\boxed{\tan^{-1} x = y \iff \tan y = x}$ $\boxed{-\frac{\pi}{2} < y < \frac{\pi}{2}}$

8. $\lim_{x\to\infty} \tan^{-1} x = \boxed{\frac{\pi}{2}}$ $\lim_{x\to-\infty} \tan^{-1} x = \boxed{-\frac{\pi}{2}}$



9. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

10. $y = \csc^{-1} x \ (|x| \ge 1) \iff \csc y = x$ and $y \in (0, \pi/2] \cup (\pi, 3\pi/2]$

$y = \sec^{-1} x \ (|x| \ge 1) \iff \sec y = x$ and $y \in [0, \pi/2) \cup [\pi, 3\pi/2)$

$y = \cot^{-1} x \ (x \in \mathbb{R}) \iff \cot y = x$, $y \in (0, \pi)$

⟨3.5⟩ Inverse Trigonometric Functions

[11] TABLE OF DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

⟨3.6⟩ INDETERMINATE FORMS & L'HOSPITAL'S RULE

[1] $\lim_{x\to 1} \frac{\ln x}{x-1}$

★ L'Hospital's Rules

Suppose f and g are differentiable and $g'(x) \neq$ near a (except possibly at a).
Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

$$\text{or} \quad \lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

Indeterminate Forms of $\boxed{\frac{0}{0}}$ or $\boxed{\frac{\infty}{\infty}}$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (is ∞